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On the additivity conjecture for the Weyl channels being covariant with respect to the maximum commutative group of unitaries

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Some new examples of quantum channels for which the infimum of the output entropy is additive under taking a tensor product of channels are given.

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A linear trace-preserving map Φ on the set of states (positive unit-trace operators) $\sigma(H)$ in a Hilbert space H is said to be a quantum channel if Φ^* is completely positive ([6]). The channel Φ is called bistochastic if $\Phi(\frac{1}{d}I_H)=\frac{1}{d}I_H$. Here and in the following we denote by d and I_H the dimension of H, $dimH=d<+\infty$, and the identity operator in H, respectively. Fix the basis $|f_j>\equiv|j>,\ 0\le j\le d-1$, of the Hilbert space H. We shall consider a special subclass of the bistochastic Weyl channels ([1, 2, 4, 5, 10]) defined by the formula ([2])

$$\Phi(x)=(1-(d-1)(r+dp))x+r\sum_{m=1}^{d-1}W_{m,0}xW_{m,0}^* \quad (1)$$

$$+p\sum_{m=0}^{d-1}\sum_{n=1}^{d-1}W_{m,n}xW_{m,n}^*,$$

$x\in\sigma(H)$, where $r,p\ge 0,\ (d-1)(r+dp)=1$ and the Weyl operators $W_{m,n}$ are determined as follows

$$W_{m,n}=\sum_{k=0}^{d-1}e^{\frac{2\pi i}{d}kn}|k+m\ mod\ d><k|,$$

$0\le m,n\le d-1$.

Consider the maximum commutative group $\mathcal{U}_d$ consisting of unitary operators

$$U=\sum_{j=0}^{d-1}e^{i\phi_j}|e_j><e_j|,$$

where the orthonormal basis (e_j) is defined by the formula

$$|e_j>=\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}e^{\frac{2\pi i}{d}jk}|k>,\ 0\le j\le d-1,$$

$\phi_j\in\mathbb{R},\ 0\le j\le d-1$. Notice that

$$<f_k|e_j>=\frac{1}{\sqrt{d}}e^{\frac{2\pi i}{d}jk},\ 0\le j,k\le d-1,$$

It implies that

$$|<f_k|e_j>|=\frac{1}{\sqrt{d}} \quad (2)$$

The bases (f_j) and (e_j) satisfying the property (2) are said to be mutually unbiased ([9]). It is straightforward to check that

$$W_{0,n}|e_j><e_j|W_{0,n}^*=|e_{j+n\ mod\ d}><e_{j+n\ mod\ d}|, \quad (3)$$

$0\le j,n\le d-1$.

It was shown in [2] that the Weyl channels (1) are covariant with respect to the group $\mathcal{U}_d$ such that

$$\Phi(UxU^*)=U\Phi(x)U^*,\ x\in\sigma(H),\ U\in\mathcal{U}_d.$$

The infimum of the output entropy of a quantum channel Φ is defined by the formula

$$\chi(\Phi)=\inf_{x\in\sigma(H)}S(\Phi(x)),$$

where $S(x)=-Tr(xlog(x))$ is the von Neumann entropy of the state $x\in\sigma(H)$. The additivity conjecture for the quantity $\chi(\Phi)$ states ([3])

$$\chi(\Phi\otimes\Psi)=\chi(\Phi)+\chi(\Psi)$$

for an arbitrary quantum channel Ψ.

Example 1. Put $r=p=\frac{q}{d^2},\ 0\le q\le 1$, then it can be shown ([1, 2, 4]) that (1) is the quantum depolarizing channel,

$$\Phi_{dep}(x)=(1-q)x+\frac{q}{d}I_H,\ x\in\sigma(H), \quad (4)$$

$$\chi(\Phi_{dep})=-(1-\frac{d-1}{d}q)\log(1-\frac{d-1}{d}q)-(d-1)\frac{q}{d}\log\frac{q}{d}.$$

□

Example 2. Put $r=\frac{1}{d}(1-\frac{d-1}{d}q),\ p=\frac{q}{d^2},\ 0\le q\le\frac{d}{d-1}$, then (1) is q-c-channel ([8]). Indeed, under the conditions given above the channel $\Phi\equiv\Phi_{qc}$ can be represented as follows

$$\Phi_{qc}(x)=(1-\frac{d-1}{d}q)E(x)+\frac{q}{d}\sum_{n=1}^{d-1}W_{0,n}E(x)W_{0,n},$$

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where

$$E(x)=\frac{1}{d}\sum_{m=0}^{d-1}W_{m,0}xW_{m,0}^*,$$

$x\in\sigma(H)$ is a conditional expectation on the algebra generated by the projections $|e_j><e_j|,\ 0\le j\le d-1$. Taking into account (3) we get

$$\Phi_{qc}(x)=\sum_{j=0}^{d-1}Tr(|e_j><e_j|x)x_j,\ x\in\sigma(H), \qquad (5)$$

where

$$x_j=(1-\frac{d-1}{d}q)|e_j><e_j|+$$

$$\frac{q}{d}\sum_{k=1}^{d-1}|e_{j+k\ mod\ d}><e_{j+k\ mod\ d}|,$$

$0\le j\le d-1$,

$$\chi(\Phi_{qc})=-(1-\frac{d-1}{d}q)\log(1-\frac{d-1}{d}q)-(d-1)\frac{q}{d}\log\frac{q}{d}.$$

□

In the present paper our goal is to prove the following theorem. We shall use the approach introduced in ([1, 2]).

Theorem. *Suppose that d is a prime number and $p\le r\le\frac{1}{d}(1-d(d-1)p)$. Then, for the channel (1) there exist d orthonormal bases $(h_j^s)_{j=0}^{d-1},\ 0\le s\le d-1$, in H such that*

$$S((\Phi\otimes\Psi)(x))\ge\chi(\Phi)+\frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{j=0}^{d-1}S(\Psi(x_j^s)),$$

$x\in\sigma(H\otimes K)$, where $x_j^s=dTr_H(|h_j^s><h_j^s|x)\in\sigma(K),\ 0\le j\le d-1$, and Ψ is an arbitrary quantum channel in a Hilbert space K.

The proof of the Theorem is based upon Theorem 2 from [2]. We shall formulate it here for the convenience.

Theorem 2 ([2]). *Let $\Phi(\rho)=(1-p)\rho+\frac{p}{d}I_H,\ \rho\in\sigma(H),\ 0\le p\le\frac{d^2}{d^2-1}$, be the quantum depolarizing channel in the Hilbert space H of the prime dimension d. Then, there exist d orthonormal bases $\{f_j^s,\ 0\le s,j\le d-1\}$ in H such that*

$$S((\Phi\otimes Id)(x))\ge-(1-\frac{d-1}{d}p)\log(1-\frac{d-1}{d}p)- \qquad (6)$$

$$\frac{d-1}{d}p\log\frac{p}{d}+\frac{1}{d^2}\sum_{j=0}^{d-1}\sum_{s=0}^{d-1}S(x_j^s),$$

where $x\in\sigma(H\otimes K),\ x_j^s=dTr_H((|e_j^s><e_j^s|\otimes I_K)x)\in\sigma(K),\ 0\le j,s\le d-1$.

Proof.

It follows from the condition $p\le r\le\frac{1}{d}(1-d(d-1)p)$ that there exists a number $\lambda,\ 0\le\lambda\le1$, such that $r=\lambda p+(1-\lambda)\frac{1}{d}(1-d(d-1)p)$. Hence the channel (1) can be represented as a convex linear combination of the following form

$$\Phi=\lambda\Phi_{dep}+(1-\lambda)\Phi_{qc},$$

where the channels Φ_{dep} and Φ_{qc} are defined by the formulae (4) and (5), respectively.

Let us define the phase damping channel Ξ by the formula ([1, 2])

$$\Xi(x)=\frac{1+(d-1)\lambda}{d}x+\frac{1-\lambda}{d}\sum_{m=1}^{d-1}W_{m,0}xW_{m,0}^*,\ x\in\sigma(H).$$

Then,

$$\Phi=\Xi\circ\Phi_{dep}.$$

The non-decreasing property of the von Neumann entropy gives us the estimation

$$S((\Phi\otimes\Psi)(x))\ge S((\Phi_{dep}\otimes\Psi)(x))=S((\Phi_{dep}\otimes Id)(\tilde{x})), \qquad (7)$$

where $\tilde{x}=(Id\otimes\Psi)(x)$. Applying Theorem 2 to the right hand side of (7) we obtain the result.

□

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